

On the interpolation of integer-valued polynomials

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Abstract

It is well known, that if polynomial with rational coefficients of degree n takes integer values in points $0, 1, \dots, n$ then it takes integer values in all integer points. Are there sets of $n + 1$ points with the same property in other integral domains? We show that answer is negative for the ring of Gaussian integers $\mathbb{Z}[i]$ when n is large enough. Also we discuss the question about minimal possible size of set, such that if polynomial takes integer values in all points of this set then it is integer-valued.

1 Introduction

Let R be integral domain and $f(x)$ be polynomial with coefficients from the quotient field of R (or any larger field). The polynomial f is called integer-valued if $f(R) \subset R$. There is much literature about integer-valued polynomials (see book [1] and links therein). Sometimes polynomials are integer-valued when some weaker conditions holds, for example: if $R = \mathbb{Z}$ then polynomial of degree not higher than n is integer-valued iff it takes integer values in points $0, 1, \dots, n$. This fact is well-known and can be proved in many ways (one of them is direct interpolation of f in above-mentioned points). Let us call set $M \subset R$ n -universal if any polynomial f of degree not higher than n such that $f(M) \subset R$ is integer-valued. The natural question is: what is the minimal cardinality of n -universal set? Obviously, it is not less than $n + 1$ for any infinite domain R . So there is special interest in studying situations when it is exactly $n + 1$. Sequence $\{a_0, a_1, \dots\}$ with elements in R is called simultaneous P-ordering if $\{a_0, a_1, \dots, a_n\}$ is an n -universal set for any n . Melanie Wood [2] showed that there is no simultaneous P-ordering in the rings of integers of imaginary quadratic fields (in particular in the ring $\mathbb{Z}[i]$ of Gaussian integers).

From now on we consider domain $\mathbb{Z}[i]$. Let us call its elements integer numbers and irreducible elements — prime numbers. Let $|n|$ be the absolute value of the complex number n , and $\|n\| = |n|^2$ — its norm. For any non-zero $n \in \mathbb{Z}[i]$ there are exactly $\|n\|$ residues (remainders) modulo n . Let $|M|$ be the cardinality of finite set M . For $M \subset \mathbb{Z}[i]$, $\alpha \in \mathbb{Z}[i] \setminus 0$, $r \in \mathbb{Z}[i]$ let us denote

$$M(r \bmod \alpha) := M \cap (r + \alpha\mathbb{Z}[i]).$$

The main results are following:

Theorem 1. *There are no n -universal sets of cardinality $n + 1$ in $\mathbb{Z}[i]$ provided n is large enough.*

Theorem 2. *There exists an n -universal set of cardinality $O(n)$ in $\mathbb{Z}[i]$.*

2 Proof of the theorem 1

Let $C = \{c_1, c_2, \dots, c_{n+1}\}$ be an n -universal set.

Definition 1. A volume of the finite set of integers $X = \{x_1, x_2, \dots, x_k\}$ is a product of all differences between elements of X :

$$V(X) = \prod_{1 \leq i < j \leq k} (x_i - x_j).^1$$

¹Formally volume is defined only up to sign, but its choice is unimportant for the later considerations.

Definition 2. Non-negative integer numbers a_1, a_2, \dots, a_N are called almost equal if any two of them are either equal or differ by one. Finite set $M \subset \mathbb{Z}[i]$ is almost uniformly distributed modulo non-zero number $\alpha \in \mathbb{Z}[i]$ if numbers $|M(r \bmod \alpha)|$ (where r runs over all possible remainders modulo α) are almost equal.

We are going to use the following standard facts:

Proposition 1. 1) There are exactly one way to split non-negative integer N into sum of k almost equal numbers.

2) This and only this partition minimizes sum of squares of summands.

3) If we split N into k almost equal parts, and each of them into ℓ almost equal parts, then we get the partition of N into kl almost equal parts.

Lemma 1. $C = \{c_1, c_2, \dots, c_{n+1}\}$ is an n -universal set if and only if C is almost uniformly distributed modulo p^k for each prime power p^k . In that case volume $V(C)$ is minimal (by absolute value) among all volumes of the sets consisting of $n+1$ integer points, and moreover $V(C)$ divides volume $V(B)$ for any other set B . If also $|V(B)| = |V(C)|$ then set B is n -universal too.

Proof. For arbitrary $c_m \in C$ let us consider the polynomial $Q_m(x)$ of degree n , which takes value 0 in all points c_i (where $i \neq m$) and value 1 in the point c_m . Let f be a polynomial of degree not greater than n . By Lagrange interpolation formula we have that

$$f(x) = \sum f(c_k)Q_k(x).$$

So C is an n -universal set if and only if all polynomials Q_m are integer-valued. Also

$$Q_m(x) = \prod_{i \neq m} \frac{(x - c_i)}{(c_m - c_i)}. \quad (1)$$

Polynomial Q_m is an integer-valued if and only if for any prime p maximal degree of p that divides numerator (where x is a fixed integer) is not less than maximal degree of p dividing denominator. Now let us fix p .

Assume that elements of C is almost uniformly distributed modulo every power of p . Then for any power p^k there are at least as many multiplies that divides p^k in the numerator as in the denominator. Indeed, there are exactly $r - 1$ such multiplies in the denominator (where $r = |C(c_m \bmod p^k)|$). And numerator has exactly $|C(x \bmod p^k)|$ such multiplies, which is not less than r under our assumption. After summing by $k = 1, 2, \dots$ one gets the above-mentioned condition on the maximal degree of p in the numerator and denominator.

Now assume that elements of C is not almost uniformly distributed p^k for some p^k , and let k be minimal among such numbers. Then for some r the set $C(r \bmod p^{k-1})$ is not almost uniformly distributed modulo p^k (the contrary would contradict to the part 3 of Proposition 1). Let $|C(c_m \bmod p^k)| \geq |C(r \bmod p^k)| + 2$ and $c_m \equiv r \bmod p^{k-1}$. Without loss of generality $|C(c_m \bmod p^{k+1})| > |C(r \bmod p^{k+1})|$ (otherwise we may replace c_m with some $c_{m'}$ and r with some r' with the same remainders modulo p^k and suitable remainders modulo p^{k+1}). Similarly we may think that

$$|C(c_m \bmod p^{k+\rho})| > |C(r \bmod p^{k+\rho})|, \rho = 1, 2, \dots$$

Consider the formula of $Q_m(r)$. If $l = 1, 2, \dots, k - 1$ then there are equal number of multiplies divisible by p^l in the numerator and denominator. If $l = k$ then denominator has more such multiplies than numerator, and if $l > k$ then not less than numerator. Summing by l we get that power of p in the numerator is less than in the denominator, so Q_m is not integer-valued.

First part of the lemma is now proved. Note that if C is almost uniformly distributed modulo $q = p^k$ then number of pairs in C with equal remainders modulo q is minimal among all $n + 1$ -element sets. Last follows immediately from the part 2 of Proposition 1. Summing by $k = 1, 2, \dots$ we get that power of p in the $V(C)$ is minimal if and only if C is almost uniformly distributed modulo p^k for any k . This proves the second and third parts of the lemma. \square

Note that for any particular p it is not hard to construct the set, which is almost uniformly distributed modulo p^k (for any k). However “to combine” all these sets into one (which fits all p) is not always possible.

For example, for $n = 1, 2, 3, 5$ there are universal sets $\{0, 1\}$, $\{0, 1, i\}$, $\{0, 1, i, 1+i\}$, $\{0, 1, 2, i, 1+i, 2+i\}$. But, as one may easily check, there is no 4-universal set with 5 elements.

Now let $C = \{c_1, c_2, \dots, c_{n+1}\}$ be an n -universal set.

Our next aim is to replace set C with more convenient set without increasing absolute value of its volume. We are going to symmetrize set C with respect to vertical and horizontal lines of integer lattice. We need the following combinatorial lemma for that.

Lemma 2. *Let A and B be some finite sets of integer rational points of prescribed cardinalities. Fix number $k \in \mathbb{N}_0$. Then number of pairs $a \in A$, $b \in B$ such that $|a - b| \leq k$ is minimal when each of sets A and B is a set of consecutive integer points and midpoints of segments formed by A and B are equal (in case of $|A| \equiv |B| \pmod{2}$) or differ by $\frac{1}{2}$ (otherwise).*

Proof. Replace each point $b \in B$ by the segment b' with length $2k$ and midpoint in b . Then $|a - b| \leq k$ exactly when $a \in b'$.

Now we are going to modify set of segments B' without decreasing the number of mentioned pairs, until segments becomes consecutive. Let the leftmost segment start in the point 0, let there also be some segments that start in the points $1, \dots, s-1$, but no segment starts in the point s . Move each of mentioned segments one to the right. Let Z_- be a set of integer points which now are covered by one less segments from B' , Z_+ — by one more segments from B' . All other points are covered by the same number of segments as before.

Then $|Z_-| = |Z_+| = \min(s, 2k+1) = z$. Denote $A_i = A \cap Z_i$, $|A_i| = m_i$, where $i \in \{-, +\}$. If $m_- \leq m_+$ then above-mentioned shift does not decrease the number of interesting pairs. Otherwise, denote $m = m_- - m_+ > 0$. Consider m leftmost points in A_- . They were covered (totally) by at most $t = (z - m_- + 1) + (z - m_- + 2) + \dots + (z - m_+)$ segments. Consider m leftmost points in $Z_+ \setminus A_+$. They are now covered (totally) by at least t segments. So, if in addition to shift we replace these m points in A with the mentioned m points in $Z_+ \setminus A_+$, then number of interesting pairs does not decrease.

If there was at least one segment in B' apart from shifted ones, then sum of pairwise distances between midpoints of all segments in B' decreased. So in the end of our process the segments in B' must be consecutive. Obviously, after that, points of A should be placed in the way described in the statement. \square

For each vertical line $\{x = c\}$ replace the points of C on this line to the segment centered either in $(c, 0)$ or $(c, 1/2)$ (depending on parity of the number of points).

Using Lemma 2 to all numbers $k \in \mathbb{N}_0$ and all pairs of vertical lines we see that this operation does not increase the absolute value of the volume. After that one may act in the same way with the horizontal lines.

Hereupon set C becomes rather “symmetric”. In particular, if n_1 is the length of the segment in the intersection C and real axis, n_2 — same for the imaginary axis, then set C is placed inside a rectangle $n_1 \times n_2$.

We need the following well-known fact for the later reasoning:

Theorem. *Quotient of two consecutive rational prime numbers of the form $4k+3$ tends to 1.*

For particular $\varepsilon > 1$ there always exists prime numbers of the form $4k+3$: $p \in (\sqrt{n}, \varepsilon\sqrt{n})$ and $q \in \left(\sqrt{\frac{n}{2}}, \varepsilon\sqrt{\frac{n}{2}}\right)$ when n is large enough. These numbers are also primes in $\mathbb{Z}[i]$. But $\|p\| > n$, so elements of C cannot give equal remainders by modulo p . In particular $n_1 < p$ and $n_2 < p$. So set C is placed inside some square $(p-1) \times (p-1)$. Moreover $2\|q\| > n$, so elements of C may give each remainder by modulo q at most twice. It is easy to see that in this case there are no

more than $p^2 - 2(p - q)^2$ points in C . So

$$\begin{aligned} n &< p^2 - 2(p - q)^2 \\ n &< -p^2 + 4pq - 2q^2 \\ p^2 + n + 2q^2 &< 4pq \\ 3n &< 2\sqrt{2}\varepsilon n \\ \frac{3}{2\sqrt{2}} &< \varepsilon. \end{aligned}$$

Taking ε small enough one gets a contradiction. Theorem 1 is now proved.

3 Examples of universal sets

Let us prove Theorem 2.

Let p be a prime number. Set C of cardinality $n + 1$ is called (n, p) -universal if it is almost uniformly distributed modulo p^k (for any k). Proof of Lemma 1 shows that if polynomial f of degree at most n takes integer values in all points of some (n, p) -universal set, then its value in any integer point is a fraction without p in the denominator. Hence set that contains (n, p) -universal subset for any p is always n -universal.

Now we are going to prove that integer points inside the circle with center in 0 and radius R contains at least cR^2 points that form (n, p) -universal set (here c is some absolute constant). From this we conclude that integer points inside the circle $\sqrt{\frac{n+1}{c}}$ forms an n -universal set as required.

Let $|p^k| \leq R\sqrt{2} < |p|^{k+1}$.

Consider all integers that lies in half-open square

$$\{p^k(a + bi), -1/2 - A \leq a, b < 1/2 + A\}$$

for some non-negative rational integer A .

Clearly each remainder modulo p^k occurs exactly $(2A + 1)^2$ times among considered points. Choose A in such a way that

- (i) considered points lies inside a circle with the center 0 and radius R ;
- (ii) no two of considered points have equal remainders by modulo p^{k+1} .

Property (i) holds if $(1/2 + A)|p^k| \leq R$, property (ii) definitely holds if diagonal of the square is shorter than p^{k+1} , i. e. if $(2A + 1)\sqrt{2}|p^k| \leq |p^{k+1}|$, or $2A + 1 \leq |p|/\sqrt{2}$. Note that $A = 0$ satisfies both conditions. Also note that two consecutive half-integer rational numbers differs in no more than three times. Choose maximal A that satisfies both inequalities. We have that

$$A + 1/2 \geq \frac{1}{3} \min(R/|p^k|, \frac{|p|}{2\sqrt{2}}) \geq \frac{R}{6|p^k|}.$$

Number of selected points is $(2A + 1)|p^k|^2 \geq R^2/9$ as required.

Theorem 2 is proved.

The following conjecture generalizes both our theorems.

Conjecture. *The minimal cardinality of the n -universal set in $\mathbb{Z}[i]$ grows as $\frac{\pi}{2}n + o(n)$ and asymptotically sharp example is realized on the set of integer points inside the circle of radius $\sqrt{n/2} + o(\sqrt{n})$.*

Note that the points inside less circle do not form n -universal set for sure (because the points inside circle of radius R give not all possible residues modulo some prime of absolute value slightly greater then $R\sqrt{2}$).

It also looks probable that the theorems, analagous to proven here for $\mathbb{Z}[i]$, hold for wide class of integral domains.

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References

- [1] Paul-Jean Cahen, Jean-Luc Chabert. *Integer-valued polynomials*. AMS, 1997.
- [2] Melanie Wood. *P-orderings: a metric viewpoint and the non-existence of simultaneous orderings*. J. Number Theory, 99 (2003), 36-56.